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Piecewise Polynomial Approximation on Optimal Meshes

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1. NOTATION AND MAIN RESULTS

A mesh of order k is a nondecreasing $(k+1)$-tuple of reals. If $u = (u_0, u_1, ..., u_k)$ is a mesh, let $\#u = k$ denote the *order* and $\lambda(u) = \max_{1 \leqslant i \leqslant k} (u_i - u_{i-1})$ the *mesh-size* of u. On occasion we adopt the view that u is a collection of open intervals $I_i = (u_{i-1}, u_i)$, called *the intervals of* u, and we write $I_i \in u$. A single open interval is a mesh of order one. A *partition* is a mesh with nonempty intervals.

If u is a mesh *on* the open interval (a, b); i.e., $u_0 = a$ and $u_{\#u} = b$, then $P^n(u)$ is the collection of real functions on (a, b) whose restrictions to the intervals of u are polynomials of degree at most $n-1$. If $f \in \mathbf{L}^p(a, b)$, $1 \leqslant p \leqslant \infty$, and u is mesh on (a, b), let

$$E_{p,n}(f, u) = \inf\{\| f - s \|_{p,(a,b)} : s \in P^n(u)\}.$$

If $\#u = 1$, $u = (\alpha, \beta) = I$, we also write $E_{p,n}(f, \alpha, \beta)$ or $E_{p,n}(f, I)$ for $E_{p,n}(f, u)$. The quantities of greatest interest to us are, for $k = 1, 2, 3, ...,$

$$E_{p,n}(f, k)_{(a,b)} = \inf\{E_{p,n}(f, u) : \#u = k, u_0 = a, u_k = b\}.$$

Let us write

$$\sigma = (n + p^{-1})$$

and with u ranging over all possible meshes on (a, b) let

$$B_{p,n}(f, u) = \sum_{I \in u} E_{p,n}(f, I)^{\sigma},$$

$$B_{p,n}(f)_{(a,b)} = \limsup_{\lambda(u) \to 0} B_{p,n}(f, u),$$

$$N_{p,n}(f)_{(a,b)} = \sup_{u} B_{p,n}(f, u).$$

Obviously, it is enough to consider partitions u in the last two equations. The interval designation (a, b) is often dropped. If f and g are in $\mathbf{L}^p(a, b)$ then, because $\sigma \leqslant 1$,

$$B_{p,n}(f + g, u) \leqslant B_{p,n}(f, u) + B_{p,n}(g, u).$$

Similarly, the functionals $B_{p,n}(\cdot)$ and $N_{p,n}(\cdot)$ satisfy the triangle inequality, and, moreover, are homogeneous of degree σ.

DEFINITION 1.1. Let $\mathbf{N}^{p,n}(a, b)$ designate the collection of elements f of $\mathbf{L}^p(a, b)$ such that $N_{p,n}(f) < \infty$. It is not hard to see that $\mathbf{N}^{p,n}(a, b)$ is a linear space, which becomes an F-space when supplied with the norm

$$|f|_{p,n} = \|f\|_p + N_{p,n}(f).$$

For $p = \infty$ replace $\mathbf{L}^p(a, b)$ by $\mathbf{C}[a, b]$. Only for $\sigma = 1$ is $\mathbf{N}^{p,n}(a, b)$ a Banach space, cf. [13], p. 51ff for the terminology and basic facts regarding F-spaces.

The Sobolev space of real functions f possessing on (a, b) an n-th distribution derivative $f^{(n)}$ in $\mathbf{L}^p(a, b)$ is denoted by $\mathbf{W}^{n,p}(a, b)$, $1 \leqslant p \leqslant \infty$ and $n \in \{1, 2, 3, ...\}$. $\mathbf{W}^{n,p,\mathrm{loc}}(a, b)$ is the collection of locally integrable real functions f on (a, b) such that $f \in \mathbf{W}^{n,p}(\alpha, \beta)$ if $a < \alpha < \beta < b$.

Main Results. We now state the main results of this paper. Consider a fixed interval (a, b), positive integer n and $1 \leqslant p \leqslant \infty$.

THEOREM 1.1. (i) *The Sobolev space* $\mathbf{W}^{n,1}(a,b)$ *is contained in* $\mathbf{N}^{p,n}(a, b)$. *Denote the closure of* $\mathbf{W}^{n,1}(a, b)$ *in the metric of* $\mathbf{N}^{p,n}(a, b)$ *by* $\mathbf{N}_0^{p,n}(a, b)$. *Then if* $f \in \mathbf{N}_0^{p,n}(a, b)$

$$\lim_{k \to \infty} k^n E_{p,n}(f, k) = B_{p,n}(f)^{1/\sigma}. \tag{1}$$

(ii) *If either* $f \in \mathbf{W}^{n,1}(a, b)$ *or else* $f \in \mathbf{W}^{n,1,\mathrm{loc}}(a, b) \cap \mathbf{L}^p(a, b)$ *and* $|f^{(n)}|$ *is monotone a.e. with* $|f^{(n)}|$ *in* $\mathbf{L}^\sigma(a, b)$, *then* $f \in \mathbf{N}_0^{p,n}(a, b)$ *and Eq.* (1) *holds with*

$$B_{p,n}(f) = c_{p,n}^{\sigma} \|f^{(n)}\|_{\sigma}^{\sigma}, \qquad c_{p,n} = E_{p,n}\left(\frac{x^n}{n!}, 0, 1\right). \tag{2}$$

(iii) *If* $f \in \mathbf{W}^{n,1,\mathrm{loc}}(a, b) \cap \mathbf{L}^p(a, b)$, *then*

$$\liminf_{k\to\infty} k^n E_{p,n}(f, k) \geqslant c_{p,n} \|f^{(n)}\|_\sigma . \tag{3}$$

Throughout, if $p = \infty$, *we assume* $f \in \mathbf{C}[a, b]$.

Part (ii) can be stated in slightly stronger form, by replacing "$|f^{(n)}|$ is monotone" by "$|f^{(n)}|$ has a locally integrable majorant in $\mathbf{L}^\sigma(a, b)$ which is monotone near a and near b." This is shown in Section 2.

THEOREM 1.2. (i) *If* $f \in \mathbf{L}^p(a, b)$ *then for each positive integer* k

$$k^n E_{p,n}(f, k) \leqslant N_{p,n}(f)^{1/\sigma}. \tag{4}$$

(ii) $N_{p,n}(f) < \infty$ *if and only if* $B_{p,n}(f) < \infty$.

(iii) *If* $f \in \mathbf{W}^{n-1,1}(a, b)$ *and* $f^{(n-1)}$ *is of bounded variation, denote the total variation of* $f^{(n-1)}$ *by* $\|f^{(n)}\|_V$. *Then with* $d_n = 1/(n-1)!$

$$N_{p,n}(f) \leqslant d_n^\sigma (b-a)^{1-\sigma} \|f^{(n)}\|_V^\sigma . \tag{5}$$

(iv) *If* $f \in \mathbf{W}^{n,1,\mathrm{loc}}(a, b)$ *and* $|f^{(n)}|$ *is monotone, then for some constant* $M_{p,n}$,

$$N_{p,n}(f) \leqslant M_{p,n} \|f^{(n)}\|_\sigma^\sigma . \tag{6}$$

Previous results on the problems of this paper were obtained by Lawson [6], Ream [9], Phillips [8], Sacks and Ylvisaker [11] and earlier work referenced therein, Rice [10], McClure [7], Burchard [1], Freud and Popov [5], Subbotin and Chernykh [12], de Boor [3], and Dodson [4]. More detailed references are given at the appropriate places below.

The proofs of Theorems 1.1 and 1.2 begin with establishing the inequalities

$$\begin{aligned} B_{p,n}(f, T^{k,1})^{1/\sigma} &\leqslant k^n E_{p,n}(f, T^{k,1}) = k^n E_{p,n}(f, k) \\ &\leqslant k^n E_{p,n}(f, T^{k,2}) = B_{p,n}(f, T^{k,2})^{1/\sigma}. \end{aligned} \tag{7}$$

Here, $T^{k,1}$ and $T^{k,2}$ are meshes of order k on (a, b), constructed in Section 3, depending on f, p, n, and k. $T^{k,1}$ is an *optimal* mesh, as is clear from (7), while $T^{k,2}$ is *balanced for* f; i.e., $E_{p,n}(f, I)$ has identical values for all intervals I of $T^{k,2}$. Theorems 1.1 and 1.2 follow by proving that the extreme terms in (7) converge or are bounded as claimed. The requisite approximation theory is in Section 2.

If $p = \infty$, then we can take $T^{k,1} = T^{k,2}$ so equality holds in (7), but in general this is not so. It is easy, e.g., to explicitly construct counterexamples for $n = 1$, $p = 1, 2$. For $p = \infty$, see [6].

The first inequality in (7) follows from Hölder's inequality. De Boor [3] and Dodson [4] used Hölder's inequality to derive a similar inequality. We have the following.

LEMMA 1.3. *For any mesh u on (a, b) and f in $\mathbf{L}^p(a, b)$*

$$B_{p,n}(f, u) \leqslant k^{n\sigma} E_{p,n}(f, u)^\sigma, \qquad k = \#u.$$

Proof. We use Hölder's inequality with dual exponents $q = 1/(n\sigma)$ and $q' = p/\sigma$, so that $1/q + 1/q' = 1$. Consider the vectors $y, z \in \mathbb{R}^k$, $k = \#u$, with components $y_i = 1$, $z_i = E_{p,n}(f, I_i)^\sigma$, $I_i = (u_{i-1}, u_i)$. Then, summing over $i = 1,\ldots, k$, for $1 \leqslant p < \infty$

$$\begin{aligned} B_{p,n}(f, u) = y \cdot z &\leqslant \left(\sum 1\right)^{n\sigma} \left(\sum E_{p,n}(f, I_i)^p\right)^{\sigma/p} \\ &= k^{n\sigma} E_{p,n}(f, u)^\sigma. \end{aligned}$$

This generalizes to $p = \infty$.

Phillips [8] showed

$$E_{p,n}(f, a, b) = c_{p,n}(b - a)^{1/\sigma} \, |f^{(n)}(\xi)| \,, \quad \xi \in [a, b], \quad \text{if} \quad f \in \mathbf{C}^n[a, b]. \tag{8}$$

The constant $c_{p,n}$ is the one that occurs in (3). We do not require this result here, except in the trivial case $f^{(n)} =$ const. but it is helpful to note that (8) implies

$$B_{p,n}(f, u) = c_{p,n}^\sigma \sum_{i=1}^{\#u} (u_i - u_{i-1}) \, |f^{(n)}(\xi_i)|^\sigma \tag{9}$$

with $u_{i-1} \leqslant \xi_i \leqslant u_i$, for $f \in \mathbf{C}^n[a, b]$. Thus, $B_{p,n}(f, u)$ is a Riemann sum and we obtain a special case of Theorem 2.8, cf. Section 2 below:

$$B_{p,n}(f) = \lim_{\lambda(u)\to 0} B_{p,n}(f, u) = c_{p,n}^\sigma \int_a^b |f^{(n)}(t)|^\sigma \, dt \quad \text{if} \quad f \in \mathbf{C}^n[a, b]. \tag{10}$$

Thus, for f in $\mathbf{C}^n[a, b]$, Theorem 2.8 (ii) below follows from (8). Phillips [8], in attempting a similar proof made the assumption that an optimal mesh is necessarily balanced. Since this is valid only for $p = \infty$, his proof is restricted to this case. (He has the additional restriction that $f|_{(\alpha,\beta)}$ not be a polynomial of degree $n - 1$ or less on any interval $(\alpha, \beta) \subset (a, b)$).

De Boor and Dodson [3], [4] have obtained asymptotic upper and lower bounds, instead of the limit in (2), of the form

$$K \, \| f^{(n)} \|_\sigma$$

assuming $f \in \mathbf{W}^{n,\infty}(a, b)$ and $f^{(n)}$ Riemann integrable. They also derive the upper bound for $|f^{(n)}|$ monotone. Previously, Burchard [1] had obtained the upper bound for $f \in \mathbf{C}^n[a, b]$ and Freud and Popov [5] and Subbotin and Chernykh [12] have the upper bound that follows from (5) and (6), except for different multiplicative constants.

Estimates for Spline Functions. Similar to $P^n(u)$ is $S^n(u)$, the spline functions of degree $n - 1$ on the mesh u. This is the subset of $P^n(u)$ described by the usual smoothness conditions: If $\alpha \in \mathbb{R}$ and α occurs with multiplicity m in the mesh u, then $s \in S^n(u)$ is to have continuous derivatives at least up to order $n - m - 1$ at $\alpha(\alpha \neq u_0, \alpha \neq u_k, k = \#u)$. With u ranging over all possible meshes on (a, b) let

$$P_k^n(a, b) = \bigcup_{\#u = k} P^n(u),$$

$$S_k^n(a, b) = \bigcup_{\#u = k} S^n(u).$$

We have the containment relations

$$P^n_{[k/n]}(a, b) \subset S_k^n(a, b) \subset P_k^n(a, b),$$

and hence the inequalities

$$(2n)^n[k/n]^n E_{p,n}(f, [k/n]) \geqslant k^n \operatorname{dist}_p(f, S_k^n(a, b)) \geqslant k^n E_{p,n}(f, k). \quad (11)$$

These inequalities demonstrate that the asymptotic behavior of spline approximation, as $k \to \infty$ on optimal meshes, can be closely estimated in terms of $E_{p,n}(f, k)$, an idea first used by Rice [10]. Theorem 1.1 shows that for $f \in \mathbf{W}^{n,1}(a, b)$ or $f \in \mathbf{L}^p(a, b)$, $|f^{(n)}|$ monotone ($f \in \mathbf{C}[a, b]$ if $p = \infty$), (11) implies

$$\begin{aligned} (2n)^n c_{p,n} \|f^{(n)}\|_\sigma &\geqslant \limsup_{k \to \infty} k^n \operatorname{dist}_p (f, S_k^n(a, b)) \\ &\geqslant \liminf_{k \to \infty} k^n \operatorname{dist}_p (f, S_k^n(a, b)) \\ &\geqslant c_{p,n} \|f^{(n)}\|_\sigma . \end{aligned} \quad (12)$$

The last inequality holds for all f in $\mathbf{W}^{n,1,\mathrm{loc}}(a, b)$.

Relationship between $B_{p,n}(f)$ and $N_{p,n}(f)$. We state here and prove separately the result of Theorem 1.2 (ii) because this is fundamental for the relevance of the seminorm $N_{p,n}(f)$ to our problems.

PROPOSITION 1.4. *If* $f \in \mathrm{L}^p(a, b)$, $1 \leqslant p \leqslant \infty$, *then* $N_{p,n}(f) < \infty$ *if and only if* $B_{p,n}(f) < \infty$.

Proof. Clearly $B_{p,n}(f) \leqslant N_{p,n}(f)$, and so assume $B_{p,n}(f) < \infty$. Then, if $x \in [a, b]$, we can find $\delta > 0$ and a finite bound M such that for all meshes u on (a, b)

$$\sum_{I \in u, I \subset (x-\delta, x+\delta)} E_{p,n}(f, I)^\sigma < M. \tag{13}$$

For suppose $x \in [a, b]$ and no such δ and M can be found. Let $\delta > 0$, $M < \infty$ and u a partition on (a, b) such that (13) is violated. It is easy to construct a partition u' on (a, b) such that: (i) if I is an open interval, $I \subset (x - \delta, x + \delta)$ then $I \in u'$ iff $I \in u$; (ii) if $I \not\subset (x - \delta, x + \delta)$ and $I \in u'$, then $\lambda(I) < 3\delta$. This implies that $\lambda(u') < 3\delta$, and $B_{p,n}(f, u') \geqslant M$. This construction may be carried out for every $\delta > 0$ and $M < \infty$, and so $B_{p,n}(f) = \limsup_{\lambda(u)\to 0} B_{p,n}(f, u) = \infty$, contrary to the hypothesis. Thus, we can assign to each x in $[a, b]$ a δ-neighborhood and $M < \infty$ such that (13) holds for all meshes u. Select a finite cover of $[a, b]$ by, say m, such neighborhoods. We obtain x_i, $\delta_i > 0$, $M_i < \infty$, $i = 1, 2,\dots, m$ such that

$$\begin{aligned} &\text{(i)} \quad [a, b] \subset \bigcup_{i=1}^{m} (x_i - \delta_i, x_i + \delta_i), \\ &\text{(ii)} \quad \sum_{I \in u, I \subset (x_i-\delta_i, x_i+\delta_i)} E_{p,n}(f, I)^\sigma < M, \end{aligned} \tag{14}$$

the latter for all meshes u on (a, b), $M = \max_{1\leqslant i\leqslant m} M_i$. Now let $\varphi(x) = \max_{1\leqslant i\leqslant m} \operatorname{dist}(x, [a, b]\backslash V_i)$, $V_i = (x_i - \delta_i, x_i + \delta_i)$. We can find a positive δ that bounds φ from below on $[a, b]$. Considering any mesh u on (a, b) we find that the intervals I of u that are not contained in some V_i have $\lambda(I) > 2\delta$. Thus there are no more than $N = [1 + (b - a)(2\delta)^{-1}]$ such intervals. Since $E_{p,n}(f, I)$ is a monotone function of I, cf. Lemma 2.9 below, the contribution of such intervals to $B_{p,n}(f, u)$ does not exceed $N \cdot M_1{}^\sigma$, where $M_1 = E_{p,n}(f, a, b)$. For the remaining intervals of u, each contained in some V_i, we have by (14) that

$$\sum_{I \in u, (\exists i) I \subset V_i} E_{p,n}(f, I)^\sigma < mM,$$

and thus we have shown $B_{p,n}(f, u) \leqslant N \cdot M_1{}^\sigma + m \cdot M$, and so $N_{p,n}(f)$ is finite.

Cross references are such that (1.7) refers to formula (7) in Section 1.

2

In this section we establish basic estimates, continuity properties and approximation theorems for the seminorms $B_{p,n}$ and $N_{p,n}$.

LEMMA 2.1. *Let $d_n = 1/(n-1)!$. If $f \in \mathbf{W}^{n-1,1}(a, b)$ with $f^{(n-1)} \in \mathbf{BV}(a, b)$ and $1 \leqslant p \leqslant \infty$ then*

$$E_{p,n}(f, a, b) \leqslant d_n(b-a)^{1/\sigma-1} \|f^{(n)}\|_V \tag{1}$$

$$N_{p,n}(f)^{1/\sigma} \leqslant d_n(b-a)^{1/\sigma-1} \|f^{(n)}\|_V . \tag{2}$$

Here, $\|f^{(n)}\|_V = \int_{(a,b)} |df^{(n-1)}|$ is the total variation of the Radon–Stieltjes measure $f^{(n)} = df^{(n-1)}$. In particular, if $f \in \mathbf{W}^{n,1}(a, b)$ then (1) and (2) are valid with $\|f^{(n)}\|_V = \|f^{(n)}\|_1$.

Proof. Repeated integration by parts establishes Taylor's formula

$$f(x) = s(x) + d_n \int_{(a,x)} (x-t)^{n-1}\, df^{(n-1)}(dt),$$

where $s^{(n)} = 0$, i.e., $s \in P^n(a, b)$. Then, for $1 \leqslant p < \infty$,

$$\begin{aligned} E_{p,n}(f, a, b) &\leqslant d_n \left[\int_a^b \left(\int_{(a,x)} |x-t|^{n-1} |df^{(n-1)}(dt)|\right)^p dx\right]^{1/p} \\ &\leqslant d_n(b-a)^{1/\sigma-1} \|f^{(n)}\|_V . \end{aligned}$$

For $p = \infty$, (1) follows similarly. Now, let u be a partition of (a, b). Then, with summation over $i = 1,..., \#u$, and using (1) on each interval of the partition, we obtain

$$\begin{aligned} B_{p,n}(f, u) &= \sum E_{p,n}(f, u_{i-1}, u_i)^\sigma \\ &\leqslant d_n^\sigma \sum (u_i - u_{i-1})^{1-\sigma} \|f^{(n)}\|^\sigma_{V,(u_{i-1},u_i)} \\ &\leqslant d_n^\sigma \left(\sum (u_i - u_{i-1})\right)^{1-\sigma} \left(\sum \|f^{(n)}\|_{V,(u_{i-1},u_i)}\right)^\sigma \\ &\leqslant d_n^\sigma(b-a) \|f^{(n)}\|_{V,(a,b)} . \end{aligned}$$

We have used Hölder's inequality. Now (2) follows by taking the supremum over all partitions u of (a, b).

DEFINITION 2.2. By Lemma 2.1 $\mathbf{W}^{n,1}(a, b)$ is contained in $\mathbf{N}^{p,n}(a, b)$. By $\mathbf{N}_0^{p,n}(a, b)$ we denote the closure of $\mathbf{W}^{n,1}(a, b)$ in $\mathbf{N}^{p,n}(a, b)$.

Lemma 2.1 also shows that $S^n(u) \subset \mathbf{N}^{p,n}$ for any partition u. The following inequality of de Boor [3] and Dodson [4] allows us to conclude that an important class of functions belongs to $\mathbf{N}_0^{p,n}$, cf. Proposition 2.5.

LEMMA 2.3. (de Boor and Dodson) *If n is a positive integer, $1 \leqslant p \leqslant \infty$, and if g is a nonnegative and non-decreasing function on (a, b), then the function*

$$\varphi(x) = \int_{(a,x)} (x - t)^{n-1} g(t)\, dt$$

satisfies

$$\| \varphi \|_p \leqslant \| g \|_\sigma / (n(np + 1)^{1/p}).$$

Here we interpret $(np + 1)^{1/p} = 1$ for $p = \infty$.

De Boor and Dodson use this lemma for obtaining upper bounds, see below for more details.

LEMMA 2.4. *Suppose $f \in \mathbf{W}^{n,1,\mathrm{loc}}(a, b)$ and $| f^{(n)} |$ possesses a monotone majorant g, in the sense that $| f^{(n)}(x)| \leqslant g(x)$ a.e. on (a, b). Let φ be defined as in Lemma 2.3 and let $M_{p,n} = (n!\,(np + 1)^{1/p})^{-\sigma}$, $1 \leqslant p \leqslant \infty$. Then*

$$E_{p,n}(f, a, b) \leqslant d_n \| \varphi \|_{p,(a,b)} \leqslant M_{p,n}^{1/\sigma} \| g \|_{\sigma,(a,b)}\,, \tag{3}$$

$$N_{p,n}(f)_{(a,b)} \leqslant M_{p,n} \| g \|_{\sigma,(a,b)}^{\sigma}\,. \tag{4}$$

In particular $f \in \mathbf{N}^{p,n}(a, b)$ if $g \in \mathbf{L}^\sigma(a, b)$.

Proof. Without loss we may assume that g is nondecreasing. Then $f \in \mathbf{W}^{n,1}(a, \beta)$ for $\beta < b$. Now (3) follows by means of Taylor's formula as in the proof of Lemma 2.1, since $| f^{(n)} | \leqslant g$ a.e. Then, for any mesh u on (a, b)

$$B_{p,n}(f, u) \leqslant M_{p,n} \sum \| g \|_{\sigma,(u_{i-1},u_i)}^{\sigma} = M_{p,n} \| g \|_{\sigma,(a,b)}^{\sigma}\,,$$

summing over $i = 1,\ldots, \# u$. Now (4) follows. If $g \in \mathbf{L}^\sigma(a, b)$, then (3) shows that $f \in \mathbf{L}^p(a, b)$ and (4) shows $N_{p,n}(f) < \infty$, hence f is in $\mathbf{N}^{p,n}(a, b)$.

Among the functions f to which Lemma 2.4 applies are such important examples as $f_\alpha(x) = x^\alpha$, $0 < x < 1$, $\alpha > -1/p$. For these J. R. Rice [10] proved $\limsup_{k\to\infty} k^n E_{p,n}(f_\alpha, k) < \infty$. The suggestion by H. G. Burchard [1] that this could be attributed to the fact that $| f_\alpha^{(n)} | \in \mathbf{L}^\sigma(0, 1)$ was carried out successfully by de Boor and Dodson, loc. cit., who made use of the monotonicity of $| f_\alpha^{(n)} |$ and proved lemma 2.3.

The results just stated are much sharpened as well as generalized in Theorems 1.1 and 1.2. These depend in part on the following proposition, which strengthens Lemma 2.4.

PROPOSITION 2.5. *We assume $f \in \mathbf{W}^{n,1,\mathrm{loc}}(a, b)$ and that $|f^{(n)}|$ has a majorant g a.e., satisfying* (a) $g \in \mathbf{L}^{1,\mathrm{loc}}(a, b)$; (*b*) $g \in \mathbf{L}^{\sigma}(a, b)$; (c) *g is monotone near a and near b. Then f is in $\mathbf{N}_0^{p,n}(a, b)$. If $p = \infty$, then f has a continuous extension to $[a, b]$. More specifically, let $\epsilon > 0$ and let $f_\epsilon \in \mathbf{W}^{n,1}(a, b)$ be defined such that*

$$
\begin{aligned}
f_\epsilon(x) &= f(x) && \text{for} \quad a - \epsilon \leqslant x \leqslant b - \epsilon, \\
f_\epsilon^{(n)}(x) &= 0 && \text{for} \quad a < x < a + \epsilon \quad \text{or} \quad b - \epsilon < x < b.
\end{aligned}
$$

Then for $1 \leqslant p \leqslant \infty$

$$
\lim_{\epsilon \to 0+} (\| f - f_\epsilon \|_p + N_{p,n}(f - f_\epsilon)) = 0.
$$

Proof. Note that f is continuous in (a, b). If g is nondecreasing near a then $|f^{(n)}|$ is actually integrable on each interval $[a, \beta]$, $\beta < b$. Similarly near b. Thus we consider the case when g is nonincreasing near a and nondecreasing near b. It follows from Lemma 2.4 that f is in $\mathbf{L}^p(a, b)$. In particular, for $p = \infty$ f is a bounded continuous function on (a, b). To avoid trivial repetitions in the arguments we slightly simplify the assertion by assuming that g, hence $|f^{(n)}|$, is actually integrable near a. Accordingly we can now more simply let f_ϵ in $\mathbf{W}^{n,1}(a, b)$ be defined such that

$$
\begin{aligned}
f_\epsilon(x) &= f(x) && \text{for} \quad a < x \leqslant b - \epsilon, \\
f_\epsilon^{(n)}(x) &= 0 && \text{for} \quad b - \epsilon < x < b.
\end{aligned}
$$

Proceeding with these simplifications in mind, abbreviate $c = b - \epsilon$, and assume $\epsilon > 0$ is sufficiently small such that g is nondecreasing on (c, b). Then by Lemma 2.4, for $1 \leqslant p \leqslant \infty$,

$$
\begin{aligned}
\| f - f_\epsilon \|_{p,(a,b)} &= \| f - f_\epsilon \|_{p,(c,b)} \leqslant d_n \| \varphi_\epsilon \|_{p,(c,b)} \\
&\leqslant M_{p,n}^{1/\sigma} \| g \|_{\sigma,(c,b)} .
\end{aligned}
$$

Here, $\varphi_\epsilon(x) = \int_{(c,x)} (x - t)^{n-1} g(t)\, dt$, $c < x < b$. The preceeding estimate shows that $f_\epsilon \to f$ in $\mathbf{L}^p(a, b)$, by the monotone convergence theorem, since $g \in \mathbf{L}^\sigma(a, b)$. For $p = \infty$ it follows that f has a continuous extension to $[a, b]$, being the uniform limit of the functions f_ϵ in $\mathbf{W}^{n,1}(a, b)$.

It remains to prove that $N_{p,n}(f - f_\epsilon)_{(a,b)} \to 0$ as $\epsilon \to 0+$. Let $g_0(x) = 0$ for $a < x \leqslant c$ and $g_0(x) = g(x)$ for $c < x < b$. Then g_0 is a monotone majorant for $|f^{(n)} - f_\epsilon^{(n)}|$ on (a, b). Hence by Lemma 2.4

$$
\begin{aligned}
N_{p,n}(f - f_\epsilon)_{(a,b)} &\leqslant M_{p,n} \| g_0 \|_{\sigma,(a,b)}^{\sigma} \\
&= M_{p,n} \| g \|_{\sigma,(c,b)}^{\sigma} .
\end{aligned}
$$

Again the monotone convergence theorem implies the right-hand side in this inequality tends to zero, and we have shown $f \in \mathbf{N}_0^{p,n}(a, b)$.

Incidentally, we have for the metric of $\mathbf{N}^{p,n}$

$$\| f - f_\epsilon \|_{p,n} \leqslant M_{p,n}^{1/\sigma} \| g \|_\sigma + M_{p,n} \| g \|_\sigma^\sigma$$

in the preceding proof.

The utility of the family of functions $\mathbf{N}_0^{p,n}(a, b)$ stems in part from the fact that we are able to prove the following lemma about approximation by splines "of one degree higher."

LEMMA 2.6. *For every $f \in \mathbf{N}_0^{p,n}(a, b)$ and $\epsilon > 0$ there exists a $\delta > 0$ such that for all partitions u with $\lambda(u) < \delta$ there is $s \in S^{n+1}(u)$ with $N_{p,n}(f - s) < \epsilon$. If $f \in \mathbf{W}^{n,1}(a, b)$ we can achieve in addition that $(b - a)^{1-\sigma} \| f^{(n)} - s^{(n)} \|_1^\sigma < \epsilon/d_n^\sigma$.*

Proof. It suffices to show the lemma for $f \in \mathbf{W}^{n,1}(a, b)$. Choose $g \in \mathbf{C}^n[a, b]$ such that $\| f^{(n)} - g^{(n)} \|_1 < \eta$, $\eta > 0$ to be chosen later. Now let u be any partition with $\lambda(u) < \delta$ where δ is chosen so that $\omega(g^{(n)}, \delta) < \eta$, $\omega(\cdot, \delta)$ being the modulus of continuity. Then construct $s \in S^{n+1}(u)$ so that, for $i = 1,..., \#u$ and $u_{i-1} < t < u_i$

$$s^{(n)}(t) = g^{(n)}(u_{i-1}).$$

Then

$$\| s^{(n)} - g^{(n)} \|_1 < \eta(b - a)$$

and so

$$\begin{aligned} \| f^{(n)} - s^{(n)} \|_1^\sigma &\leqslant (\| f^{(n)} - g^{(n)} \|_1 + \| g^{(n)} - s^{(n)} \|_1)^\sigma \\ &< \eta^\sigma(1 + b - a)^\sigma. \end{aligned}$$

Hence, by (2),

$$N_{p,n}(f - s) \leqslant d_n^\sigma(b - a)^{1-\sigma} \eta^\sigma(1 + b - a)^\sigma = \phi(\eta).$$

Finally, choose $\eta > 0$ so that $\phi(\eta) < \epsilon$ and the proof is completed.

The norm $B_{p,n}(s)$ is easily evaluated for $s \in S^{n+1}(u)$.

LEMMA 2.7. *For partitions u and v on (a, b) such that each interval of v is contained in some interval of u and for $s \in S^{n+1}(u)$*

$$B_{p,n}(s) = B_{p,n}(s, v) = c_{p,n}^\sigma \| s^{(n)} \|_\sigma^\sigma ,$$

where $c_{p,n} = E_{p,n}(x^n, 0, 1)/n!$. Notice that $| s^{(n)} |^\sigma$ is piecewise continuous, hence Riemann integrable.

Proof. The result (1.8) of Phillips [8], in the trivial case when $f^{(n)} =$ const. gives for $s \in S^{n+1}(a, b)$ and all $c \in \mathbb{R}$

$$E_{p,n}(s, a, b) = (b - a)^{1/\sigma} c_{p,n} \mid s^{(n)}(c)\mid .$$

on any interval (a, b). Hence if $s \in S^{n+1}(v)$

$$B_{p,n}(s, v) = \sum E_{p,n}(s, v_{i-1}, v_i)^\sigma$$

$$= \sum c_{p,n}^\sigma(v_i - v_{i-1}) \mid s^{(n)}(c_i)\mid^\sigma$$

$$= c_{p,n}^\sigma \| s^{(n)} \|_\sigma^\sigma ,$$

where $v_{i-1} < c_i < v_i$ and $i = 1,..., \#v$.

THEOREM 2.8. (i) *Suppose $f \in \mathbf{N}_0^{p,n}(a, b)$. Then*

$$\lim_{\lambda(u)\to 0} B_{p,n}(f, u) = B_{p,n}(f). \tag{5}$$

(ii) *If f is either in $\mathbf{W}^{n,1}(a, b)$ or else satisfies the hypotheses of Proposition* 2.5 *then, more precisely,*

$$\lim_{\lambda(u)\to 0} B_{p,n}(f, u) = c_{p,n}^\sigma \| f^{(n)} \|_\sigma^\sigma = B_{p,n}(f), \tag{6}$$

($c_{p,n}$ as in Lemma 2.7).

(iii) *Finally, if $f \in \mathbf{W}^{n,1,\mathrm{loc}}(a, b) \cap \mathbf{L}^p(a, b)$, then*

$$\liminf_{\lambda(u)\to 0} B_{p,n}(f, u) \geqslant c_{p,n}^\sigma \| f^{(n)} \|_\sigma^\sigma .$$

Proof. Let $\epsilon > 0$ and choose $\delta > 0$ as in Lemma 2.6. If $\lambda(u) < \delta$ for a partition u find $s \in S^{n+1}(u)$ such that $N_{p,n}(f - s) < \epsilon/2$. Then

$$\mid B_{p,n}(f) - B_{p,n}(f, u)\mid \leqslant \mid B_{p,n}(f) - B_{p,n}(s)\mid + \mid B_{p,n}(s, u) - B_{p,n}(f, u)\mid$$

$$\leqslant B_{p,n}(f - s) + B_{p,n}(f - s, u)$$

$$\leqslant 2N_{p,n}(f - s) < \epsilon.$$

We have shown (5).

Next assume $f \in \mathbf{W}^{n,1}(a, b)$. Now, if $\delta > 0$, $\lambda(u) < \delta$ and $s \in S^{n+1}(u)$ are as above, then by Lemmas 2.6 and 2.7

$$\begin{aligned} &| B_{p,n}(f) - c^{\sigma}_{p,n} \| f^{(n)} \|^{\sigma}_{\sigma} | \\ &\quad \leqslant | B_{p,n}(f) - B_{p,n}(s)| + | c^{\sigma}_{p,n} \| s^{(n)} \|^{\sigma}_{\sigma} - c^{\sigma}_{p,n} \| f^{(n)} \|^{\sigma}_{\sigma} | \\ &\quad \leqslant N_{p,n}(f - s) + c^{\sigma}_{p,n} \| s^{(n)} - f^{(n)} \|^{\sigma}_{\sigma} \\ &\quad \leqslant \epsilon/2 + c^{\sigma}_{p,n}(b - a)^{1-\sigma} \| s^{(n)} - f^{(n)} \|^{\sigma}_{1} < \epsilon/2 + (\epsilon c^{\sigma}_{p,n}/2d_n{}^{\sigma}) < \epsilon. \end{aligned}$$

We made use of Hölder's inequality: $\| g \|^{\sigma}_{\sigma} \leqslant (b - a)^{1-\sigma} \| g \|^{\sigma}_{1}$, and of the obvious $c_{p,n} \leqslant d_n$. The inequalities above hold for any $\epsilon > 0$, hence

$$B_{p,n}(f) = c^{\sigma}_{p,n} \| f^{(n)} \|^{\sigma}_{\sigma}$$

holds for $f \in \mathbf{W}^{n,1}(a, b)$, and thus (5) implies (6) in this case.

Next, let f satisfy the hypotheses of Proposition 2.5. For $\epsilon > 0$ sufficiently small the function $f_\epsilon(x)$ defined there has, by (6),

$$B_{p,n}(f_\epsilon) = c^{\sigma}_{p,n} \| f^{(n)}_{\epsilon} \|^{\sigma}_{\sigma,(a,b)} = c^{\sigma}_{p,n} \int_{a+\epsilon}^{b-\epsilon} | f^{(n)} |^{\sigma}\, dt.$$

Hence

$$\begin{aligned} &| B_{p,n}(f) - c^{\sigma}_{p,n} \| f^{(n)} \|^{\sigma}_{\sigma} | \\ &\quad \leqslant | B_{p,n}(f) - B_{p,n}(f_\epsilon)| + c^{\sigma}_{p,n} | \| f^{(n)} \|^{\sigma}_{\sigma} - \| f^{(n)}_{\epsilon} \|^{\sigma}_{\sigma} | \\ &\quad \leqslant N_{p,n}(f - f_\epsilon) + c^{\sigma}_{p,n} \| f^{(n)} - f^{(n)}_{\epsilon} \|^{\sigma}_{\sigma}. \end{aligned}$$

The first term on the right tends to zero as $\epsilon \to 0+$ by Proposition 2.5, the second term by the monotone convergence theorem. Thus, since $f \in \mathbf{N}^{p,n}_{0}(a, b)$ by Proposition 2.5, we have shown (5) and (6) in the present case.

Finally, consider the case when $f \in \mathbf{W}^{n,1,\mathrm{loc}}(a, b) \cap \mathbf{L}^{p}(a, b)$. If

$$M < c^{\sigma}_{p,n} \| f^{(n)} \|^{\sigma}_{\sigma},$$

choose $\epsilon > 0$ so that

$$c^{\sigma}_{p,n} \| f^{(n)} \|^{\sigma}_{\sigma,(a+\epsilon,b-\epsilon)} > M,$$

and $\delta > 0$ so that, if w is a partition of $(a + \epsilon, b - \epsilon)$, then $\lambda(w) < \delta$ implies

$$| B_{p,n}(f, w) - c^{\sigma}_{p,n} \| f^{(n)} \|^{\sigma}_{\sigma,(a+\epsilon,b-\epsilon)} | < \epsilon.$$

Now, if u is a partition of (a, b) with $\lambda(u) < \delta$ let w be that partion of $(a + \epsilon, b - \epsilon)$ the intervals of which are the ones of u, intersected with $(a + \epsilon, b - \epsilon)$.

Clearly $\lambda(w) < \delta$, since it is easy to see that $E_{p,n}(f, \alpha, \beta)$ depends monotonically on the interval (α, β). Therefore,

$$B_{p,n}(f, u) \geq B_{p,n}(f, w) \geq c^{\sigma}_{p,n} \| f^{(n)} \|^{\sigma}_{\sigma,(a+\epsilon, b-\epsilon)} - \epsilon \geq M - \epsilon.$$

This proves (iii). Note that we have not assumed $\| f^{(n)} \|_{\sigma} < \infty$.

The monotonicity property of $E_{p,n}(f, \alpha, \beta)$ used in the preceding proof is stated as part of the next lemma, needed below for frequent reference.

LEMMA 2.9. *Let $a \leq \alpha < \beta \leq b$. $E_{p,n}(f, \alpha, \beta)$ depends continuously on (α, β) for f in $\mathbf{L}^p(a, b)$, if $1 \leq p < \infty$, and for f in $\mathbf{C}[a, b]$ if $p = \infty$. Furthermore $E_{p,n}(f, \alpha, \beta)$ is nonincreasing in α and nondecreasing in β. For $p = \infty$ and $f \in \mathbf{C}[a, b]$,*

$$E_{\infty,n}(f, \alpha, \beta) \leq \omega(f, |\beta - \alpha|).$$

The proof is straightforward and is omitted.

We now proceed to establish a companion result of Theorem 2.8 that greatly strengthens it and that is needed below in the proof of Theorem 1.1; however, it is well to point out that Theorem 2.8 is all that is required if Theorem 1.1 is specialized slightly by restricting to functions f in $\mathbf{L}^p(a, b)$ such that $f|_{(\alpha,\beta)}$ is never a polynomial of degree $n - 1$ or less for

$$a \leq \alpha < \beta \leq b.$$

The extension of Theorem 2.8 that is needed has to do with the following "weighted" mesh size, which measures the subintervals I of a partition by the distance of f from $P^n(I)$.

DEFINITION 2.10. Let u be a mesh and assume $f \in \mathbf{L}^p(u_0, u_k)$, $k = \#u$. Define

$$\mu_{p,n}(u, f) = \max_{1 \leq i \leq k} E_{p,n}(f, u_{i-1}, u_i),$$

the "mesh size of u weighted by f." If no confusion can arise, we also write simply $\mu(\cdot)$ for $\mu_{p,n}(\cdot, f)$.

The relationship between $\lambda(u)$ and $\mu_{p,n}(u, f)$ is as follows.

LEMMA 2.11. *Suppose $f \in \mathbf{L}^p(a, b)$, $1 \leq p < \infty$, or $f \in \mathbf{C}[a, b]$, $p = \infty$, and let $\mu(u, f) = \mu_{p,n}(u, f)$ for partitions u of (a, b). Then*

$$\lim_{\lambda(u) \to 0} \mu(u, f) = 0. \tag{7}$$

$$\lim_{\mu(u,f) \to 0} \lambda(u) = 0 \qquad \textit{if and only if} \quad f|_{(\alpha,\beta)} \notin P^n(\alpha, \beta) \quad \textit{for all intervals} \quad (\alpha, \beta) \subset (a, b). \tag{8}$$

The straightforward proof is again omitted.

According to this lemma, if $f \in \mathbf{L}^p(a, b)$ and f has "trivial intervals" (α, β); i.e., $f|_{(\alpha,\beta)} \in P^n(\alpha, \beta)$ and $\alpha < \beta$, then a sequence of partitions $u^1, u^2, \ldots$ of (a, b) may satisfy $\lim_{k\to\infty} \mu_{p,n}(u^k, f) = 0$ but $\lambda(u^k)$ may fail to converge to zero. Then Theorem 2.8 does not allow one to conclude that

$$\lim_{k\to\infty} B_{p,n}(f, u^k) = B_{p,n}(f),$$

even if $f \in \mathbf{N}_0^{p,n}(a, b)$. This conclusion is nevertheless valid, as we now show.

THEOREM 2.12. *Suppose $f \in \mathbf{L}^p(a, b)$, $1 \leqslant p < \infty$, or $f \in \mathbf{C}[a, b]$, $p = \infty$, and $\lim_{\lambda(u)\to 0} B_{p,n}(f, u) = B_{p,n}(f)$ for meshes u of (a, b). If (u^k) is a sequence of meshes with $\mu_{p,n}(u^k, f) \to 0$ as $k \to \infty$, then*

$$\lim_{k\to\infty} B_{p,n}(f, u^k) = B_{p,n}(f).$$

Proof. A *trivial* interval (α, β) of f is one for which $E_{p,n}(f, \alpha, \beta) = 0$. Clearly each such interval is contained in a maximal open trivial interval. The maximal open trivial intervals of f can be arranged in a sequence $(V_i)_{i=1}^\infty$, such that $\lim_{i\to\infty} \lambda(V_i) = 0$. We write

$$V_i = (\alpha_i, \beta_i), \qquad i = 1, 2, \ldots.$$

It could happen that $V_i = \varnothing$ for all $i = 1, 2, \ldots$. Now, let (u^k) be a sequence of meshes on the interval (a, b), write $\mu(\cdot) = \mu_{p,n}(\cdot, f)$, and assume

$$\lim_{k\to\infty} \mu(u^k) = 0.$$

Choose a sequence m_k of positive integers such that

$$m_1 \leqslant m_2 \leqslant \cdots \leqslant m_k \leqslant m_{k+1} \leqslant \cdots,$$

$$\lim_{k\to\infty} m_k = \infty, \tag{9}$$

$$\lim_{k\to\infty} (m_k \cdot \mu(u^k)^\sigma) = 0. \tag{10}$$

For each $k = 1, 2, \ldots$ define a new mesh $u^{k,1}$ by adjoining to u^k the extra knots $\alpha_1, \ldots, \alpha_{m_k}, \beta_1, \ldots, \beta_{m_k}$. Further expand each $u^{k,1}$ to a mesh $u^{k,2}$ by adding knots in the trivial intervals V_j, $1 \leqslant j \leqslant m_k$, and in such a manner that no knot in V_j is farther than $1/k$ from either neighbor. Notice that for each k

$$B_{p,n}(f, u^{k,1}) = B_{p,n}(f, u^{k,2}) \tag{11}$$

since $u^{k,2}$ differs from $u^{k,1}$ only by knots added in the trivial intervals of $u^{k,1}$.

We claim now that

$$\lim_{k\to\infty} \lambda(u^{k,2}) = 0 \tag{12}$$

and for $k = 1, 2,\ldots$

$$| B_{p,n}(f, u^{k,1}) - B_{p,n}(f, u^k)| \leqslant 2m_k \cdot \mu(u^k)^\sigma. \tag{13}$$

This shown, the hypothesis of the theorem in conjunction with (10) will imply that $\lim_{k\to\infty} B_{p,n}(f, u^{k,1}) = B_{p,n}(f)$. But then the conclusion of the theorem follows from (10) and (13).

Next, we first show (12). Without loss, assume that $\lim_{k\to\infty} \lambda(u^{k,2})$ exists, otherwise select a suitable subsequence. Then let (a_k, b_k) be an interval of $u^{k,2}$ such that

$$b_k - a_k = \lambda(u^{k,2}).$$

It is understood that a_k and b_k are neighbors in $u^{k,2}$. By selecting a further subsequence, if necessary, we can assume that $a_k \to a_0$ and $b_k \to b_0$ as $k \to \infty$. Since we assume $\mu(u^k) \to 0$, then also $\mu(u^{k,2}) \to 0$, and hence, by Lemma 2.9,

$$E_{p,n}(f, a_0, b_0) = \lim_k E_{p,n}(f, a_k, b_k) = 0.$$

Hence we can find j, $1 \leqslant j < \infty$ such that $(a_0, b_0) \subset V_j = (\alpha_j, \beta_j)$. Since $\lim_k m_k = \infty$, α_j and β_j belong to $u^{k,2}$ for $k \geqslant k_0$. Now if $\alpha_j = \beta_j$ then $b_0 - a_0 = 0$ and (12) is shown. Otherwise $\alpha_j < \beta_j$ and by construction of $u^{k,2}$ neighbors in V_j of $u^{k,2}$ differ by at most $1/k$ (this is preserved under selecting subsequences), for $k \geqslant k_0$. Since $\alpha_j \leqslant a_0 \leqslant b_0 \leqslant \beta_j$ it is impossible to have $a_0 < b_0$, so $b_0 - a_0 = 0$, showing (12).

Finally, we prove (13). Recall that by construction

$$u^{k,1} = u^k \cup \{\alpha_1, \ldots, \alpha_{m_k}, \beta_1, \ldots, \beta_{m_k}\}.$$

Then clearly

$$\mu(u^{k,1}) \leqslant \mu(u^k) \tag{14}$$

and (13) will follow by induction over m_k if it can be verified for $m_k = 1$. Now, in the latter case, at most two subintervals of u^k are disturbed in passing to $u^{k,1}$. If one subinterval, I, then this is represented by three subintervals I_1, I_2, I_3 in $u^{k,1}$ and I_2 is trivial. Furthermore $E_{p,n}(f, I_l) \leqslant E_{p,n}(f, I)$ for $l = 1, 3$ and thus

$$\left| E_{p,n}(f, I)^\sigma - \sum_{l=1}^{3} E_{p,n}(f, I_l)^\sigma \right| \leqslant 2\mu(u^k)^\sigma,$$

whence (13) with $m_k = 1$. If, however, two subintervals I, J of u^k are disturbed in passing to $u^{k,1}$, then four new intervals I_1, I_2, J_1, J_2 appear in $u^{k,1}$, two of which are trivial, and moreover $E_{p,n}(f, I_l) \leqslant E_{p,n}(f, I)$, $E_{p,n}(f, J_l) \leqslant E_{p,n}(f, J) (l = 1, 2)$, and so in this case

$$\left|(E_{p,n}(f, I)^\sigma + E_{p,n}(f, J)^\sigma) - \sum_{l=1}^{2} (E_{p,n}(f, I_l)^\sigma + E_{p,n}(f, J_l)^\sigma)\right| \leqslant 2\mu(u^k)^\sigma,$$

whence again (13) with $m_k = 1$. Thus (13) has been verified for $m_k = 1$. By induction it follows easily for integers $m_k \geqslant 1$. because of (14).

3

We now construct the sequences of meshes $(T^{k,1})_{k=1}^{\infty}$ and $(T^{k,2})_{k=1}^{\infty}$, depending on f in $\mathbf{L}^p(a, b)$, for which the relations (1.7) obtain. Combining this with the results of Section 2 we obtain the proofs of Theorems 1.1 and 1.2.

LEMMA 3.1. *For $f \in \mathbf{L}^p(a, b)$, $1 \leqslant p < \infty$ or $f \in \mathbf{C}[a, b]$, $p = \infty$ and each $k = 1, 2, \ldots$, there is a mesh $T^{k,1}$ such that $\#T^{k,1} = k$ and*

$$E_{p,n}(f, T^{k,1}) = E_{p,n}(f, k), \qquad k = 1, 2, \ldots; \tag{1}$$

$$\lim_{k \to \infty} \mu(T^{k,1}, f) = 0. \tag{2}$$

Proof. By the elements of real analysis, $\lim_{k\to\infty} E_{p,n}(f, k) = 0$. Now, cf. Definition 2.10,

$$\mu_{p,n}(T^{k,1}, f) \leqslant E_{p,n}(f, T^{k,1})$$

and so (1) implies (2). To prove the existence of meshes $T^{k,1}$ satisfying (1), note that $E_{p,n}(f, u)$ is a continuous function on the compact set

$$\{u \in \mathbb{R}^{k-1} : a \leqslant u_1 \leqslant \cdots \leqslant u_{k-1} \leqslant b\},$$

by Lemma 2.9, and thus attains its minimum, at some mesh $T^{k,1}$.

The next lemma establishes the existence of meshes $T^{k,2}$. These are similar to partitions used by Burchard [1] for $f \in \mathbf{C}^n[a, b]$, and to partitions used by de Boor and Dodson [3, 4]. For $f \in \mathbf{C}^n[a, b]$ Burchard used meshes u for which

$$\int_{(u_{i-1}, u_i)} \max(|f^{(n)}|^\sigma, \eta) \tag{3}$$

(with small positive η) does not depend on $i = 1, \ldots, \#u$. For $f \in \mathbf{C}^n[a, b]$ he showed that these partitions allow to obtain upper bounds for

$$\limsup_{k\to\infty} k^n E_{p,n}(f, k) \text{ for } 0 < p \leqslant \infty$$

involving $\|f^{(n)}\|_\sigma$. For $f \in \mathbf{W}^{n,\sigma}(a, b)$ (and $f^{(n)}$ Riemann integrable) de Boor and Dodson used meshes such that

$$(u_i - u_{i-1})^{1/\sigma} \| f^{(n)} \|_{\infty,(u_{i-1},u_i)} \tag{4}$$

does not depend on i. The meshes $u = T^{k,2}$, to be constructed now, are such that

$$E_{p,n}(f, u_{i-1}, u_i) \tag{5}$$

does not depend on i. Relationships between the quantities (3), (4), and (5) are not yet entirely clear. While (5) can be bounded by (4), with a similar inequality in the opposite direction, cf. Phillips' result (1.8) and also de Boor and Dodson [3, 4], no such relationship exists between (3) and (5), as is to be shown in a future paper. We have not yet investigated to what extent properties of the meshes $T^{k,2}$ are shared by the meshes obtained by "balancing" (3) or (4).

LEMMA 3.2. *If $f \in \mathbf{L}^p(a, b)$ for $1 \leqslant p < \infty$, or $f \in \mathbf{C}[a, b]$ for $p = \infty$, there is for each positive integer k a mesh $T^{k,2}$ on (a, b) such that $\#T^{k,2} = k$, $T^{k,2}$ is balanced, and*

$$E_{p,n}(f, T_{i-1}^{k,2}, T_i^{k,2}) = \mu_{p,n}(T^{k,2}, f) \qquad \text{for} \quad i = 1,..., k. \tag{6}$$

$$\lim_{k\to\infty} \mu_{p,n}(T^{k,2}, f) = 0. \tag{7}$$

$$B_{p,n}(f, T^{k,2})^{1/\sigma} = k^n E_{p,n}(f, T^{k,2}). \tag{8}$$

Proof. Suppose for the moment that (6) holds. Then for $1 \leqslant p < \infty$

$$\mu_{p,n}(T^{k,2}, f)^p = E_{p,n}(f, T^{k,2})^p/k \leqslant \|f\|_p^p/k,$$

so (7) follows. For $p = \infty$ argue like this: For suitable subscript i, $1 \leqslant i \leqslant k$,

$$\begin{aligned}\mu_{\infty,n}(T^{k,2}, f) &= E_{p,n}(f, T_{i-1}^{k,2}, T_i^{k,2}) \leqslant \omega(f, | T_{i-1}^k - T_i^k |) \\ &\leqslant \omega\left(f, \frac{b-a}{k}\right),\end{aligned}$$

cf. Lemma 2.9. This shows (7) for $p = \infty$. Next, since $1 - (\sigma/p) = n\sigma$ we have for $1 \leqslant p < \infty$, if (6) holds,

$$B_{p,n}(f, T^{k,2}) = k[E_{p,n}(f, T^{k,2})^p/k]^{\sigma/p} = k^{n\sigma} E_{p,n}(f, T^{k,2})^\sigma,$$

and (8) follows. For $p = \infty$, when $n = 1/\sigma$, (8) is an immediate consequence of (6).

To show the existence of meshes $T^{k,2}$ satisfying (6), we inductively construct meshes

$$w^k(\beta) = (a, w^k(\beta)_1, w^k(\beta)_2, ..., w^k(\beta)_{k-1}, \beta)$$

with $w^k(\beta)_0 \equiv a$, $w^k(\beta)_k \equiv \beta$, and such that

$$E_{p,n}(f, w^k(\beta)_{i-1}, w^k(\beta)_i) = \mu_{p,n}(w^k(\beta), f) \underset{\text{def}}{=} F_k(\beta)$$
$$\text{for} \quad i = 1, 2, ..., k. \tag{9}$$

This done, we let

$$T^{k,2} = w^k(b),$$

and the proof is completed.

The construction of $w^k(\beta)$ is as follows. Having proved the existence of the meshes $w^{k-1}(\beta)$ for $a \leqslant \beta \leqslant b$ we obtain $w^k(\beta)$ in the form

$$w^k(\beta) = (a, w^{k-1}(\alpha)_1, w^{k-1}(\alpha)_2, ..., w^{k-1}(\alpha)_{k-2}, \alpha, \beta), \tag{10}$$

where α is the smallest solution of the equation

$$F_{k-1}(\alpha) - E_{p,n}(f, \alpha, \beta) = 0. \tag{11}$$

This equation ensures that $w^k(\beta)$ as given in (10) has the property (9). To show the existence of a smallest solution

$$\alpha = A_k(\beta)$$

we demonstrate by induction simultaneously

$$A_k(\beta) \text{ exists and is nondecreasing in } \beta; \tag{12}$$

$$F_k(\alpha) \text{ is continuous and nondecreasing in } \alpha: \tag{13}$$

for $k = 1, 2,...$. For $k = 1$, $A_1(\beta) \equiv a$ and $F_1(\alpha) = E_{p,n}(f, a, \alpha)$, and so (12), (13) follow from Lemma 2.9. If now l is an integer, $l \geqslant 2$, assume (12), (13) are shown for $k = l - 1$. Next, let $k = l$. The existence of $A_k(\beta)$ is then clear from (13) and Lemma 2.9 since these imply the left hand side in (11) is continuous in α for $a \leqslant \alpha \leqslant \beta$, and clearly nonpositive for $\alpha = a$ and nonnegative for $\alpha = \beta$. Let $\alpha = A_k(\beta)$ in (10). This defines $w^k(\beta)$ and $F_k(\beta)$, and these satisfy (9). Notice that for $a \leqslant \beta \leqslant b$

$$F_k(\beta) = F_{k-1}(A_k(\beta)) = E_{p,n}(f, A_k(\beta), \beta). \tag{14}$$

To show $A_k(\beta)$ is nondecreasing, assume to the contrary that there are β and $\delta > 0$ and $A_k(\beta + \delta) < A_k(\beta)$. Abbreviate

$$G(\alpha, \beta) = E_{p,n}(f, \alpha, \beta).$$

Then we have the following string of relations, consequences of (14), Lemma 2.9 and induction:

$$\begin{aligned} F_k(\beta + \delta) &= F_{k-1}(A_k(\beta + \delta)) \leqslant F_{k-1}(A_k(\beta)) = G(A_k(\beta), \beta) \\ &\leqslant G(A_k(\beta + \delta), \beta) \leqslant G(A_k(\beta + \delta), \beta + \delta) \\ &= F_k(\beta + \delta). \end{aligned}$$

Clearly, we have equality throughout, and so $A_k(\beta)$ is not the smallest solution of (11). Hence $A_k(\beta)$ is nondecreasing. This in turn gives: If $a \leqslant \beta < \beta + \delta \leqslant b$, then

$$F_k(\beta) = F_{k-1}(A_k(\beta)) \leqslant F_{k-1}(A_k(\beta + \delta)) = F_k(\beta + \delta),$$

so F_k is a nondecreasing function.

We next show that F_k is continuous from the right, omitting a similar argument for left continuity. Consider a decreasing sequence $\beta_1, \beta_2, \ldots$ with limit β and write $\alpha_j = A_k(\beta_j)$. Then (α_j) is nonincreasing, bounded below by a and so converging to some $\hat{\alpha}$. Notice that $A_k(\beta) \leqslant \hat{\alpha}$. For each j

$$F_k(\beta_j) = F_{k-1}(\alpha_j) = G(\alpha_j, \beta_j).$$

Hence by induction

$$\lim_{j \to \infty} F_k(\beta_j) = F_{k-1}(\hat{\alpha}) = G(\hat{\alpha}, \beta). \tag{15}$$

By (14), and since $A_k(\beta) \leqslant \hat{\alpha}$,

$$\begin{aligned} G(\hat{\alpha}, \beta) &\leqslant G(A_k(\beta), \beta) = F_k(\beta) = F_{k-1}(A_k(\beta)) \\ &\leqslant F_{k-1}(\hat{\alpha}) = G(\hat{\alpha}, \beta), \end{aligned}$$

so that (15) shows

$$\lim_{j \to \infty} F_k(\beta_j) = F_{k-1}(\hat{\alpha}) = F_k(\beta),$$

establishing right-continuity of F_k. The proof of left continuity is similar. This concludes the inductive proof of (12) and (13) and thus of the lemma.

Proofs of Theorems 1.1 *and* 1.2. These are now immediate from Theorems 2.8 and 2.12 and Lemmas 1.3, 2.1, 3.1, and 3.2.

Remark. An example showing that $\mathbf{N}^{p,n}(a, b) \not\subseteq \mathbf{L}^p(a, b)$ is in [1].

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